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Automorphism Groups of Small (3,3)-homogeneous Logics

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Abstract We prove that (3,3)-homogeneous logics with 15 or 17 atoms are unique and for 16 atoms such logic does not exist. We also describe the automorphism groups for such logics and construct (3,3)-homogeneos logics (with 19 atoms) with either commutative automorphism group or with unique automorphism.

Keywords Quantum logic · OMP · (3,3)-homogeneous logic · Atoms of logic · Automorphism group of logic · State space · Commutative automorphism groups

Let L be a finite quantum logic (orthomodular poset, OMP), $\mathcal{A}$ be the set of all atoms in L, $\mathcal{B}$ be the set of all maximal, with respect to inclusion, orthogonal sets of atoms (called as *blocks*), $S_2(L)$ be the set of all two-valued states of L. Let m, n be natural numbers. An OMP L is called (m,n)-*homogeneous* ((m,n)-hom.), if any atom of it is contained in m blocks, and every block of L is n-element one. It is easy to see [7] that $m \cdot \operatorname{card} \mathcal{A} = n \cdot \operatorname{card} \mathcal{B}$. If $S_2(L) \neq \emptyset$ and $f \in S_2(L)$, then $\operatorname{card} \mathcal{A} = nk$, $\operatorname{card} \mathcal{B} = mk$, where $k = \operatorname{card}(f^{-1}(1) \cap \mathcal{A})$.

The well known concrete logics of the form $\mathcal{X}(k,s) = \{X \subset \{1,\dots,ks\} \mid \operatorname{card} X \equiv 0 \pmod k\}$ [5] are good examples of homogeneous OMPs. (3,3)-homogeneous logics arise when we consider a relational OMPs [3] on a finite set. Automorphisms of concrete logics are investigated, e.g. in [4].

Let us enumerate the atoms of (3,3)-hom. logic L by the natural numbers $1, 2, \dots, \operatorname{card} \mathcal{A}$ and for each block $\{i, j, n\}$ use the abbreviation $i - j - n$. Obviously every such OMP has following 7 *initial* blocks: $B_1, \dots, B_7$: 1–2–3, 1–4–5, 1–6–7, 2–8–9, 2–10–11, 3–12–13, 3–14–15. Then we construct the logics $H_2(17)$ $L_1(19)$, $H_4(19)$. We add to $B_1, \dots, B_7$ the following 10 blocks for $H_2(17)$:

4–8–15, 4–10–13, 5–9–12, 5–14–16, 6–8–16, 6–11–12, 7–9–17, 7–10–14,
11–15–17, 13–16–17.

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We add to $B_1, \ldots, B_7$ the following 12 blocks for $L_1(19)$:

$$4–8–12,\ 4–10–14,\ 5–9–16,\ 5–11–17,\ 6–9–14,\ 6–13–17,$$
$$7–11–19,\ 7–12–18,\ 8–15–19,\ 10–16–18,\ 13–16–19, 15–17–18.$$

We add to $B_1, \ldots, B_7$ the following 12 blocks for $H_4(19)$:

$$4–8–12,\ 4–10–14,\ 5–9–13,\ 5–11–16,\ 6–8–15,\ 6–11–17,$$
$$7–9–18,\ 7–10–19,\ 12–16–18,\ 13–17–19,\ 14–17–18, 15–16–19.$$

The state spaces of some (3,3)-hom. logics of the form $H_k(m)$ (with card $\mathcal{A} = m$ and k pure states) is studied, e.g. in [6].

1 The Small (3,3)-homogeneous Logics

Theorem 1 1) *Let L be an* (3,3)*-hom. logic with* 15 *atoms. Then L is unique and isomorphic to* $\mathcal{X}(2,3)$. 2) *There is no* (3,3)*-hom. logic with* 16 *atoms.*

Proof 1. Let L be an (3,3)-hom. logic with 15 atoms. Let us represent the initial blocks $B_i \in L (i = 1, \ldots, 7)$ as in Fig. 1 at the left.

In order to construct the other blocks we should keep in mind that L should not have loops of order 3 [1, 2]. We assume that the second block from atom 5 connects atoms 9 and 13. Then the third block from atom 5 connects atoms from $\{10, 11\}$ and $\{14, 15\}$. We assume without loss of generality that it is a block 5–11–15. Now, obviously, the third block from 13 connects atom 10 and one of $\{6, 7\}$. So we have a loop of order 6: $l_6 = \{1–4–5,$ 5–11–15, 15–14–3, 3–12–13, 13–10–7, 7–6–1$\}$.

Now the third block from atom 15 should connect atom 8 with either 6-th or 7-th atom. It can be only 7–8–15, as the another block 6–8–15 leads to the loop of order 3 from atom 7. All other blocks are restored uniquely: 4–10–14, 6–11–12, 4–8–12, 6–9–14 etc. So we get hypergraph as in Fig. 1 at the right.

When all 15 blocks are constructed we can easily write all two-valued states on L. There exist six s_n, $n = 1, \ldots, 6$; in the table the empty cell means that the two-valued state equal 0:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
s_1	1								1	1		1			1
s_2			1	1			1		1		1				
s_3		1			1		1					1		1	
s_4		1		1		1							1		1
s_5	1							1			1		1	1	
s_6			1		1	1		1		1					

The table show that $S_2(L)$ is full. The total representation of L as a concrete logic has two-element atoms. So L is isomorphic to logic $\mathcal{X}(2,3)$.

2. Let (3,3)-hom. logic L have 16 atoms. Let us represent the initial blocks $B_i \in L$ $(i = 1, \ldots, 7)$ as in Fig. 2 at the left. Let us call the pairs of initial blocks $\{B_2, B_3\}$, $\{B_4, B_5\}$, $\{B_6, B_7\}$ as *first, second and third layers*: L_1, L_2, L_3. Note that any not initial block can not connect atoms in the same layer. The layers are mutually equivalent with respect to atom 16 while the other blocks are not constructed. So we can suppose that the first block from 16 connects atom 11 and 15. If the second block from 16 connects atoms from L_2 and L_3 then the third block from 16 should connect two atoms of L_1. So the second block from 16

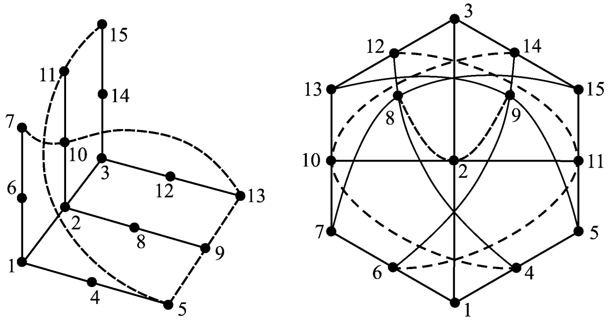

Fig. 1 (3,3)-hom. logic with 15 atoms

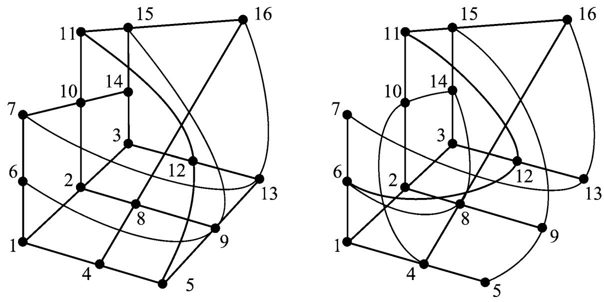

Fig. 2 Variants for (3,3)-hom. logic with 16 atoms

connects atom of L_1 with atom of L_2 or atom of L_3. Since at the current stage of construction L_2 and L_3 are equivalent we can suppose that second block is 4–8–16. Now the third block from 16 can connect only atom of $\{12, 13\}$ with atom of $\{6, 7\}$. For example, let us the third block be 7–13–16.

Now the third block from 11 can connect atom 12 with 5 or 6 only. We have two variants: 5–11–12 or 6–11–12. In the first case (see Fig. 2 at the left) the third block from 15 is restored uniquely: 6–9–15. There is the unique case (7–10–14) for third block from 7. There is the unique case (5–9–13) for third block from 5. But there is no third block from atom 4 not resulting in a loop of order 3.

Similarly, we consider the second variant (see Fig. 2 at the right). The third block from 15 is unique: 5–9–15; the third block from 6 is also unique: 6–8–14. And at last the third block from 14 must be only 4–10–14. But there is no third block from atom 5 not resulting in a loop of order 3. This completes the proof. □

Theorem 2 *Any* (3,3)-*hom. logic with* 17 *atoms is unique and isomorphic to* $H_2(17)$. $H_2(17)$ *has two pure states and has no two-valued states.*

Proof Let L be (3,3)-hom. logic with 17 atoms and initial blocks B_i $(i = 1, \dots, 7)$ be made from 15 atoms $\{1, \dots, 15\}$. There are two cases for L: 1) atoms 16 and 17 lay in a same block; 2) atoms 16 and 17 do not lay in a same block.

(1) As at the current stage of construction all layers with respect to pair $\{16, 17\}$ are equivalent we can assume that the block containing them contains also an atom of L_3: 13–16–17 (see Fig. 3).

Let us suppose that an atom of $\{16, 17\}$ do not lay in the common block with an atom of $\{14, 15\}$. This means that the all others blocks from atoms 16, 17 contain atoms from L_1 and L_2 (see Fig. 3 at the left).

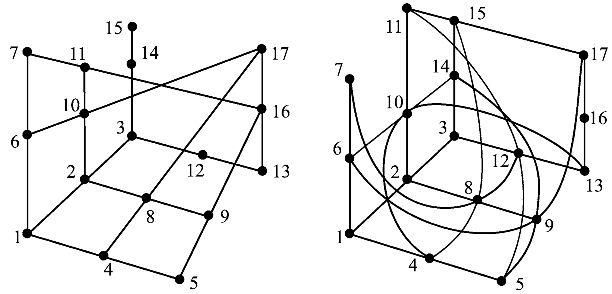

Fig. 3 Case 1) for (3,3)-hom. logic with 17 atoms

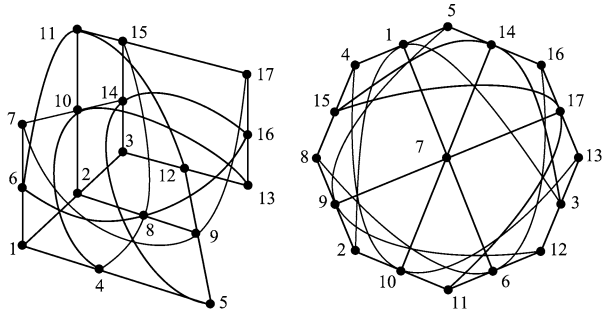

Fig. 4 (3,3)-hom. logic with 17 atoms

But there is no third block from atom 13 not resulting in a loop of order 3. So, there is a loop of order 4 in L_3. We can suppose that this loop is completed by the block 11–15–17 (see Fig. 3 at the right). Then the third block from 15 connects an atom of $\{8, 9\}$ with some atom of L_1. We can suppose that this block is 4–8–15. Now the third block from 11 necessarily connects atom 12 and an atom of $\{5, 6, 7\}$. As at the current stage of construction atoms 6 and 7 are equivalent it is suffices to consider two cases: a) 5–11–12; b) 6–11–12.

(a) The third block from 17 necessarily connects 9 and an atom of $\{6, 7\}$. We can suppose that it is 6–9–17.

Then the third block from 12 is uniquely 7–8–12 and the third block from 9 is only 5–9–14. Finally the third blocks from 13 and from 6 are 4–10–13 and 6–10–14. So, for atom 7 there is only one free atom 16, hence the third block from 7 can not be constructed. It is the contradiction.

(b) Assume that the third block from 11 is 6–11–12 (see Fig. 4 at the left). The third block from 17 necessarily connects 9 and an atom of $\{5, 7\}$. The existence of block 5–9–17 is not possible because in this case there is no third block from 12. It hence there exists block 7–9–17. Now we restore uniquely blocks 5–9–12, 4–10–13, 5–14–16. It is easy to restore the third block from 8 (6–8–16) and the last block 7–10–14 from the remaining free atoms 6, 7, 8, 10, 14, 16.

Now having the list of all the blocks of logic L, we find a loop of maximal order. This loop has order 8 (see Fig. 4 at the right).

(2) Similarly we show that when atoms 16, 17 are not lay in the same block the (3,3)-hom. logic with 17 atoms is isomorphic to $H_2(17)$.

Solving the system of 17 equations $s(i) + s(j) + s(k) = 1$, where i–j–k lists all the blocks of $H_2(17)$ we get the following: state s takes value t on atoms 1, 9, 11, 16; takes value $\frac{2}{3} - t$ on atoms 2, 5, 6, 17 and $\frac{1}{3}$ on other atoms, where $0 \leq t \leq \frac{2}{3}$. Thus $H_2(17)$ has two pure states: for $t = 0$ and for $t = \frac{2}{3}$. From this we conclude that $H_2(17)$ admits no two-valued state. The proof is complete. □

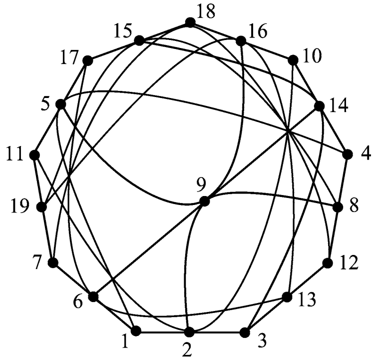

Fig. 5 (3,3)-hom. logic $L_1(19)$

2 Automorphism Groups of Small (3,3)-homogeneous Logics

Automorphism groups of $\mathcal{X}(k,s)$-logic are investigated, e.g. in [5]. It is shown that every automorphism of such a logic is uniquely determined by a bijection of the set $\{1,2,\dots,ks\}$. So this gives the description of automorphism group of (3,3)-hom. logic with 15 atoms.

Let L be a finite quantum logic, $\mathcal{A}$ be the set of all atoms in L, $\mathcal{B}$ be the set of all blocks of L. As every element of E is supremum of some $\perp$-set of atoms then automorphism of L can be identified with the bijection $\sigma:\mathcal{A}\to\mathcal{A}$ such that for all $a,b\in\mathcal{A}$ and $B\in\mathcal{B}$ the following conditions are hold:

$$(1)\quad a\perp b \Leftrightarrow \sigma(a)\perp\sigma(b);$$

$$(2)\quad \sigma(B)\in\mathcal{B}.$$

Let $a\in\mathcal{A}$. The set $a^\perp=\{b\in\mathcal{A}: b\perp a\}$ is called *orthostar* of atom a. Obviously, $\sigma(a^\perp)=(\sigma(a))^\perp$. We denote

$$R_k^m=\{a\in\mathcal{A}:\exists A\subset\mathcal{A}\,(\text{card}\,A=k)\forall b\in A\,(\text{card}(a^\perp\cap b^\perp)=m)\}$$

for $k,m\in\mathbf{N}\cup\{0\}$.

Note that for some m, k sets R_k^m may be empty; if m is fixed then these sets form a partition of $\mathcal{A}$.

Lemma 3 *Let bijection $\sigma:\mathcal{A}\to\mathcal{A}$ be an automorphism of L. Then $\sigma: R_k^m\to R_k^m$ for all k and for every fixed m.*

Proof Let $a\in R_k^m$, $\sigma(a)\in R_s^m$, $s<k$. We write down all orthostars intersecting with $a^\perp$ by m atoms: $\text{card}(a^\perp\cap b_1^\perp)=\dots=\text{card}(a^\perp\cap b_k^\perp)=m$. Similarly, let $c_1,\dots,c_s$ be all atoms such that $\text{card}(\sigma(a)^\perp\cap c_1^\perp)=\dots=\text{card}(\sigma(a)^\perp\cap c_s^\perp=m$. As σ is a bijection then $\sigma(a^\perp\cap b_\alpha^\perp)=\sigma(a)^\perp\cap\sigma(b_\alpha)^\perp$ and $\text{card}(\sigma(a)^\perp\cap\sigma(b_\alpha)^\perp)=m$ for all $\alpha=1,\dots,k$. So $\sigma(b_1),\dots,\sigma(b_k)\in\{c_1,\dots,c_s\}$ and $k\le s$. This contradiction completes the proof.

The $R_k^m\cap R_n^s$ intersections give us *main partition* of all atoms $\mathcal{A}=\cup R_j$, $R_i\cap R_j=\emptyset$. Similarly for every automorphism σ we have $\sigma:R_i\to R_i$. We denote the identity automorphism by $\mathbf{I}$. □

Theorem 4 *There exists a* (3,3)*-hom. logic with unique state and with unique automorphism* (*for example* $L_1(19)$).

Proof Note that the intersections of orthostars of hom. logic with 3 element blocks contain no more then 3 elements. So m in partitions R_k^m can take only four values: 0, 1, 2, 3. After we write all orthostars of $L_1(19)$ (see Fig. 5) and make the list of all intersections $i^\perp \cap j^\perp$, $i, j = 1, \ldots, 19$ we find all partitions ($m = 1, 2, 3$; partition for $m = 0$ does not exist):

$$R_{10}^1 = \{1, 5, 13, 14\},\ R_8^1 = \{6, 7, 10, 15, 16, 18\},\ R_9^1 = \mathcal{A}\backslash(R_8^1 \cup R_{10}^1);$$
$$R_4^2 = \{1\},\ R_7^2 = \{8\},\ R_5^2 = \{5, 13, 14\},\ R_8^2 = \{6, 7, 10, 15, 16\};$$
$$R_6^2 = \mathcal{A}\backslash(R_4^1 \cup R_5^1 \cup R_7^2 \cup R_8^2);$$
$$R_4^3 = \{1\},\ R_2^3 = \{6, 7, 8, 10, 15, 16, \},\ R_3^3 = \mathcal{A}\backslash(R_2^3 \cup R_4^3).$$

So the main partition is: $R_1 = \{1\}$, $R_2 = \{8\}$, $R_3 = \{18\}$, $R_4 = \{5, 13, 14\}$, $R_5 = \{6, 7, 10, 15, 16\}$, $R_6 = \mathcal{A}\backslash(\bigcup_{i=1}^5 R_i)$.

Let σ be an automorphism of $L_1(19)$. We denote by $\sigma_n = \sigma(n)$. By Lemma 3 we have $\sigma_1 = 1, \sigma_8 = 8, \sigma_{18} = 18$ and $\sigma_5 \in \{5, 13, 14\}$. If $\sigma_5 = 13$, then from $\{4\} = 1^\perp \cap 5^\perp$ it followes that $\{\sigma_4\} = \sigma_(1)^\perp \cap \sigma_(5)^\perp = 1^\perp \cap 13^\perp = \{3, 6\}$. So we arrive to contradiction. Similarly, the assumption $\sigma_5 = 14$ results in the false equality: $\{\sigma_4\} = 1^\perp \cap 14^\perp = \{3, 4, 6\}$. So, $\sigma_5 = 5$.

As σ transforms block to block, then $\sigma(1 - 4 - 5) = \sigma_1 - \sigma_4 - \sigma_5 = 1 - \sigma_4 - 5$. So $\sigma_4 = 4$ and $\sigma_{12} = 12$, $\sigma_7 = 7$, $\sigma_6 = 6$. Then $\sigma_2 \in \sigma_8^\perp \cap \sigma_1^\perp = \{2, 4\}$. As σ is bijection and $\sigma_4 = 4$, then $\sigma_2 = 2$. Now all the other values of the automorphism are restored uniquely: $\sigma_i = i$. Thus σ is identity.

Solving the system of linear equations $s(i) + s(j) + s(k) = 1$, $i - j - k \in \mathcal{B}$ we find the unique solution $s(i) = 1/3$ for all $i = 1, \ldots, 19$. This completes proof. □

Theorem 5 *The group* $AutH_2(17)$ *has* 12 *elements and is generated by the following involutive automorphisms* α, β, γ; β *commutes with* α *and* γ. *The cycles of these permutations are*:

$$\alpha:\quad (1)(2,5)(3,4)(6)(7)(8,12)(9)(10,14)(11,16)(13,15)(17);$$
$$\beta:\quad (1,17)(2,16)(3,13)(4,15)(5,11)(6,9)(7)(8)(10,14)(12);$$
$$\gamma:\quad (1,5)(2,16)(3,14)(4)(6,9)(7,12)(8)(10,13)(11,17)(15).$$

We need the following lemmas.

Lemma 6 *Let* σ *be an automorphism of logic* $H_2(17)$. *Then* $\sigma(17) = 17$ *if and only if either* $\sigma = \mathbf{I}$ *or* $\sigma = \alpha$.

Proof The main partition of atoms is $R_1 = \{1, 2, 5, 11, 16, 17\}$ and $R_2 = \mathcal{A}\backslash R_1$. Let $\sigma_{17} = 17$. Since $\{9, 11\} = 2^\perp \cap 17^\perp$, then $\{\sigma_9, \sigma_{11}\} = \sigma_2^\perp \cap 17^\perp$. So $\sigma_2 \in \{2, 3, 4, 5\}$. By Lemma 3 $\sigma_2 \in \{2, 5\}$.

Let $\sigma_2 = 2$. Then $\{\sigma_9, \sigma_{11}\} = 2^\perp \cap 17^\perp = \{9, 11\}$. Again by Lemma 3 $\sigma_9 = 9$, $\sigma_{11} = 11$, hence $\sigma_{10} = 10$, $\sigma_8 = 8$, $\sigma_{15} = 15$ and so on. Thus $\sigma = \mathbf{I}$.

Let now $\sigma_2 = 5$. Then $\{\sigma_9, \sigma_{11}\} = 5^\perp \cap 17^\perp = \{9, 16\}$. By Lemma 3 $\sigma_9 = 9$, $\sigma_{11} = 16$, hence $\sigma_{10} = 14$, $\sigma_8 = 12$. Thus $\{7\} = 9^\perp \cap 17^\perp$. So $\{\sigma_7\} = 9^\perp \cap 17^\perp = \{7\}$ and $\sigma_7 = 7$. Similarly, $\{\sigma_{15}\} = 16^\perp \cap 17^\perp = \{13\}$ and $\sigma_{15} = 13$. As $7 - 10$ is transformed to $7 - 14$, then $\sigma_{14} = 10$. Similarly, $15 - 14 \mapsto 13 - 10$, hence $\sigma_3 = 4$. Also $3 - 2 \mapsto 4 - 5$, hence $\sigma_1 = 1$. Now from $1 - 7 \mapsto 1 - 7$ we conclude that $\sigma_6 = 6$. From $6 - 11 \mapsto 6 - 16$ we get $\sigma_{12} = 8$. Finally from $12 - 9 \mapsto 8 - 9$ it follows that $\sigma_5 = 2$. So $\sigma^2(5) = 5$. Thus we have $\sigma^2 = \mathbf{I}$, hence σ is an involution. It allows us to restore all other values of automorphism σ and to conclude that $\sigma = \alpha$. This completes proof. □

Lemma 7 *Let σ be an automorphism of $H_2(17)$. Then the following statements are true*:

(1) $\sigma(17) = 16 \Leftrightarrow \sigma = \alpha\gamma$ *or* $\sigma = \alpha\gamma\alpha$,
(2) $\sigma(17) = 11 \Leftrightarrow \sigma = \gamma$ *or* $\sigma = \gamma\alpha$,
(3) $\sigma(17) = 5 \Leftrightarrow \sigma = \beta\gamma$ *or* $\sigma = \beta\gamma\alpha$,
(4) $\sigma(17) = 2 \Leftrightarrow \sigma = \gamma\alpha\gamma$ *or* $\sigma = \beta\alpha\gamma$,
(5) $\sigma(17) = 1 \Leftrightarrow \sigma = \beta$ *or* $\sigma = \alpha\beta$.

Proof The sufficiency of statements is obvious. We prove necessity. First we check that the involutive bijections β and γ are automorphisms of logic $H_2(17)$ and β commutes with α and γ.

(2) Let $\sigma(17) = 11$. Then $\gamma\sigma(17) = 17$. By Lemma 6 either $\gamma\sigma = \mathbf{I}$ or $\gamma\sigma = \alpha$. As $\gamma^2 = \mathbf{I}$, we have $\sigma = \gamma$ or $\sigma = \gamma\alpha$.

(5) Let $\sigma(17) = 1$. Then $\alpha\beta\sigma(17) = 17$. By Lemma 6 either $\alpha\beta\sigma = \mathbf{I}$ or $\alpha\beta\sigma = \alpha$. As α and β are involutions, then $\sigma = \beta$ or $\sigma = \alpha\beta$.

(4) Let $\sigma(17) = 2$. Then $\gamma\alpha\gamma\sigma(17) = 17$. By Lemma 6 either $\gamma\alpha\gamma\sigma = \mathbf{I}$ or $\gamma\alpha\gamma\sigma = \alpha$. Thus either $\sigma = \gamma\alpha\gamma$ or $\sigma = \gamma\alpha\gamma\alpha$. Directly we check that $\gamma\alpha\gamma\alpha = \beta\alpha\gamma$.

Similarly we prove statements (1) and (3). Lemma is proved. □

Proof of Theorem 5 Let σ be an automorphisms of logic $H_2(17)$. By Lemma 6 $\sigma : R_1 \to R_1$, when $R_1 = \{1, 2, 5, 11, 16, 17\}$. So $\sigma(17) \in R_1$. All the possible variants were considered in Lemma 6 and Lemma 7. Directly we check that the operations of multiplication and inversion do not leave the following list of automorphisms:

$$\alpha,\ \beta,\ \gamma,\ \alpha\beta,\ \beta\gamma,\ \alpha\gamma,\ \gamma\alpha,\ \alpha\gamma\alpha,\ \gamma\alpha\gamma,\ \alpha\beta\gamma,\ \beta\gamma\alpha,\ \mathbf{I}.$$

Thus the group $Aut\,H_2(17)$ is the set consisting precisely of these 12 automorphisms. Theorem is proved. □

Theorem 8 *The group* $AutH_4(19)$ *is commutative*, *has* 4 *elements and is generated by involutive automorphism* σ *and automorphism* a *with cycles*:

$$\sigma:\quad (1,2)\,(3)\,(4,8)\,(5,9)\,(6,10)\,(7,11)\,(12)\,(13)\,(14,15)\,(16,18)\,(17,19);$$

$$a:\quad (1,17,2,19)\,(3,13)\,(4,18,8,16)\,(5,14,9,15)\,(6,11,10,7)\,(12).$$

Proof The proof is similar to one of Theorem 5. □

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